

The Dynamical Fine Structure of Iterated Cosine Maps and a Dimension Paradox

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ABSTRACT. We discuss in detail the dynamics of maps $z \mapsto ae^z + be^{-z}$ for which both critical orbits are strictly preperiodic. The points which converge to ∞ under iteration contain a set R consisting of uncountably many curves called “rays”, each connecting ∞ to a well-defined “landing point” in \mathbb{C} , so that every point in \mathbb{C} is either on a unique ray or the landing point of finitely many rays.

The key features of this paper are the following two: (1) this is the first example of a transcendental dynamical system where the Julia set is all of \mathbb{C} and the dynamics is described in detail using symbolic dynamics; and (2) we get the strongest possible version (in the plane) of the “dimension paradox”: the set R of rays has Hausdorff dimension 1, and each point in $\mathbb{C} \setminus R$ is connected to ∞ by one or more disjoint rays in R ; as a complement of a 1-dimensional set, $\mathbb{C} \setminus R$ has of course Hausdorff dimension 2 and full Lebesgue measure.

1. Introduction

The dynamics of iterated polynomials is today a fairly mature subject, after three decades of activity by many people, building on the pioneering work of Douady and Hubbard. Given a polynomial p of degree $d \geq 2$, the most important set is the Julia set J consisting of points $z \in \mathbb{C}$ which have no neighborhood in which the family of iterates forms a normal family in the sense of Montel. Specifically for polynomials, one can equivalently start with the set I of points which

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converge to infinity under iteration (the *escaping* points); the complement $K = \mathbb{C} \setminus I$ is known as the *filled-in Julia set* and consists of the points with bounded orbits. Then $J = \partial I = \partial K$.

The most important case is when J and equivalently K are connected. Then there is a conformal isomorphism $\varphi: (\mathbb{C} \setminus K) \rightarrow (\mathbb{C} \setminus \overline{\mathbb{D}})$ which conjugates the dynamics of P on $I = \mathbb{C} \setminus K$ to the dynamics of $z \mapsto z^d$ on $\mathbb{C} \setminus \overline{\mathbb{D}}$. The goal is to show that the inverse Riemann map $\psi = \varphi^{-1}: \mathbb{C} \setminus \overline{\mathbb{D}} \rightarrow I$ extends continuously to the boundaries as a continuous surjection $\psi: \partial\mathbb{D} \rightarrow J$; this map would provide a topological semiconjugacy between the dynamics of z^d on $\partial\mathbb{D}$ to the dynamics of p on J . The set I is canonically foliated into *dynamic rays* $R_\vartheta = \psi((1, \infty)e^{2\pi i\vartheta})$ for $\vartheta \in \mathbb{R}/\mathbb{Z}$. The dynamic ray R_ϑ *lands* at $z \in J$ if the limit $\lim_{r \searrow 1} \psi(re^{2\pi i\vartheta})$ exists and equals z . The statement that ψ extends continuously to $\partial\mathbb{D}$ means that every dynamic ray lands, and the landing points depend continuously on the angle. By Carathéodory's theorem, this is true if and only if J is locally connected. In this case, every point $z \in J$, together with its dynamics, is described by which dynamic rays land at z , and this provides a complete description of the topological properties of the dynamics of p on J .

For transcendental entire functions f , the set I of escaping points is equally important as for polynomials, but it is much harder to understand: the set I is never empty, and it is never a neighborhood of infinity (because ∞ is an essential singularity), so there is no Riemann map providing convenient coordinates; but we still have $J = \partial I$ [E]. In many cases, I has no interior and $J = \overline{I}$. Eremenko [E] has asked whether every (path) component of I was unbounded. This has been settled in the affirmative only for the cases of exponential maps λe^z [SZ1] and for the cosine family $ae^z + be^{-z}$ [RS]: in both cases, every path component of I consists of a single curve which terminates at ∞ , and whose other end might or might not land in \mathbb{C} ; if it does, the landing point might or might not escape. Only for these rather special maps is it currently known that the escaping points are organized in the form of dynamic rays. For a larger class of maps, the existence of some curves consisting of escaping points was shown in [DT]. A description of escaping points for a large class of entire function is currently work in progress by Rottenfußer.

In this paper, we provide the first case of transcendental entire functions in which it is known that the escaping points are organized in the form of dynamic rays, such that every dynamic ray lands and every point in \mathbb{C} (which equals the Julia set) is either on a dynamic ray

or a landing point of at least one dynamic ray. To our knowledge, this is the first example of a transcendental function for which the Julia set equals \mathbb{C} and the dynamics on all of \mathbb{C} is described in terms of symbolic dynamics.

In addition, we obtain a surprising result about the Hausdorff dimensions of the involved sets: it turns out that the union of all dynamic rays has dimension 1, while each of the remaining points in \mathbb{C} (almost all points in \mathbb{C} in a very strong sense!) is the landing point of one or several rays (each ray of course has dimension 1, and so does their union!).

In [K], Karpińska had shown that for exponential maps $z \mapsto \lambda \exp(z)$ with attracting fixed points (in particular, $0 < \lambda < 1/e \subset \mathbb{R}$), the set of landing points of R has Hausdorff dimension 2, while R has dimension 1. This was extended in [SZ1] to arbitrary exponential maps: not all rays land, and not all rays which land have escaping landing points, but the union of all rays still has dimension 1, while the set of escaping landing points has dimension 2 (but planar measure zero). In [RS], the analogous result was shown for arbitrary maps $z \mapsto ae^z + be^{-z}$, except that the escaping landing points of R have even positive measure (using McMullen's result [M]). The example given in the present paper is maximal possible in the plane. In [S], our results are discussed and illustrated in special cases, together with an introductory discussion of Hausdorff dimension and background from complex dynamics.

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2. Notation and Background

Define the sets $\mathbb{Z}_L := \{\dots, -2_L, -1_L, 0_L, 1_L, 2_L, \dots\}$ and $\mathbb{Z}_R := \{\dots, -2_R, -1_R, 0_R, 1_R, 2_R, \dots\}$ (two disjoint copies of \mathbb{Z}) and set $\mathcal{S}_0 := (Z_L \cup Z_R)^{\mathbb{N}}$. To simplify notation, numbers s_L and s_R (with $s \in \mathbb{Z}$) are treated just like ordinary integers in arithmetic operations such as $2\pi i s_L = 2\pi i s$ or $|s_R| = |s|$, etc..

The space \mathcal{S}_0 is endowed with the shift map $\sigma: \mathcal{S}_0 \rightarrow \mathcal{S}_0$. For every $\underline{s} = s_1 s_2 s_3 \dots \in \mathcal{S}_0$, define

$$t_{\underline{s}} := \inf \left\{ t > 0: \lim \frac{|s_k|}{F^{\circ k}(t)} = 0 \right\} \in \mathbb{R}_0^+ \cup \{\infty\}.$$

Define $F: \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ via $F(t) := e^t - 1$. A sequence $\underline{s} = s_1 s_2 s_3 \dots \in \mathcal{S}_0$ is called *exponentially bounded* if there is an $x \in \mathbb{R}$ such that $|s_k| \leq$

$F^{\circ(k-1)}(x)$ for all k . By [SZ1, Theorem 4.2 (1)], a sequence $\underline{s} \in \mathcal{S}_0$ is exponentially bounded if and only if $t_{\underline{s}} < \infty$. Let $\mathcal{S} \subset \mathcal{S}_0$ be the space of all exponentially bounded sequences.

Define $E(z) := ae^z + be^{-z}$ with $a, b \in \mathbb{C}^*$ and $I := \{z \in \mathbb{C} : E^{\circ k}(z) \rightarrow \infty \text{ as } k \rightarrow \infty\}$ (the set of escaping points).

The following two theorems are the main results in [RS], and they hold for every map $E(z) = ae^z + be^{-z}$ with $ab \neq 0$ (if one or both critical orbits escape, then the statements have to be modified slightly in a natural way).

THEOREM 1 (Existence of dynamic rays).

Suppose that no critical orbit escapes. Then for every exponentially bounded $\underline{s} \in \mathcal{S}$ there exists a unique injective curve $g_{\underline{s}} : (t_{\underline{s}}, \infty) \rightarrow I$ consisting of escaping points such that

- (1) $g_{\underline{s}}(t) = t - \alpha + 2\pi i s_1 + o(1) \quad \text{as } t \rightarrow \infty, \text{ if } s_1 \in \mathbb{Z}_R$
- (2) $g_{\underline{s}}(t) = -t + \beta + 2\pi i s_1 + o(1) \quad \text{as } t \rightarrow \infty, \text{ if } s_1 \in \mathbb{Z}_L$
- (3) $E(g_{\underline{s}}(t)) = g_{\sigma(\underline{s})}(F(t)) \quad \text{for all } t > t_{\underline{s}}.$

Moreover, for every $t > t_{\underline{s}}$ the orbit of $g_{\underline{s}}(t)$ satisfies the following asymptotics as $k \rightarrow \infty$:

$$(4) \quad E^{\circ k}(g_{\underline{s}}(t)) = \begin{cases} F^{\circ k}(t) - \alpha + 2\pi i s_{k+1} + o(1) & \text{if } s_{k+1} \in \mathbb{Z}_R \\ -F^{\circ k}(t) + \beta + 2\pi i s_{k+1} + o(1) & \text{if } s_{k+1} \in \mathbb{Z}_L \end{cases}.$$

In particular, the orbit $z_k := E^{\circ k}(g_{\underline{s}}(t))$ satisfies

$$(5) \quad \frac{\log^+ |\text{Im}(z_k)|}{\log |\text{Re}(z_k)|} \longrightarrow 0.$$

THEOREM 2 (Escaping points are organized in rays).

Suppose that no critical orbit escapes. Then for every escaping point w there exists a unique exponentially bounded external address \underline{s} and a unique potential $t \geq t_{\underline{s}}$ such that exactly one of the following holds:

- either $t > t_{\underline{s}}$ and $w = g_{\underline{s}}(t)$,
- or $t = t_{\underline{s}}$ and the dynamic ray $g_{\underline{s}}$ lands at w such that w and the ray $g_{\underline{s}}$ escape uniformly.

In particular, every path component of I in \mathbb{C} is a dynamic ray, possibly together with the escaping landing point of the ray.

We will often need the union of all dynamic rays:

$$R := \bigcup_{\underline{s} \in \mathcal{S}} g_{\underline{s}}((t_{\underline{s}}, \infty)) \subset I.$$

The following lemma is shown in [RS].

LEMMA 3 (Horizontal Expansion).

For every $a, b \in \mathbb{C}^*$ and $h > 0$, there is an $\eta > 0$ with the following property: if (z_k) and (w_k) are two orbits under E such that $|\text{Im}(z_k) - \text{Im}(w_k)| < h$ for all k , and if $|\text{Re}(z_1)| - |\text{Re}(w_1)| > \eta$, then $z_1 \in R$: there is an $\underline{s} \in \mathcal{S}$ and a $t > t_{\underline{s}}$ such that $z_1 = g_{\underline{s}}(t)$.

While the proof is technically unpleasant, it has a very simple idea: if $|\text{Re}(z_1)| > |\text{Re}(w_1)| + \eta$, then the formula $|E(z)| \approx c \exp(|\text{Re}(z)|)$ (with $c \in \{|a|, |b|\}$) shows that $|z_2| \gg |w_2|$, and since z_2 and w_2 have essentially equal imaginary parts, this means that $|\text{Re}(z_2)| \gg |\text{Re}(w_2)|$. By induction, this shows that the real parts of (z_k) grow much faster than those of other points with comparable imaginary parts. This implies that z_1 cannot be the landing point of a dynamic ray: if $z_1 = g_{\underline{s}}(t_{\underline{s}})$ for some $\underline{s} \in \mathcal{S}$, then z_1 escapes much faster than other points on the same ray $g_{\underline{s}}$, and this leads to a contradiction to the asymptotics in Theorem 1. Therefore, Theorem 2 implies that z_1 is a point on a dynamic ray.

3. Landing of Dynamic Rays

From now on, we will restrict to the special case of *postsingularly preperiodic* maps: those maps $z \mapsto E(z) := ae^z + be^{-z}$ for which the two critical values $\pm 2\sqrt{ab}$ are strictly preperiodic. The easiest such maps are $z \mapsto k\pi \sinh(z)$ (with $k \in \mathbb{Z} \setminus \{0\}$) for which the critical values are $\pm k\pi i$, and both map to the repelling fixed point 0 (the maps $z \mapsto k\pi \sin(z)$ are the same maps in a rotated coordinate system). Slightly more generally, if $a = -b$ is such that $a(1 - \sinh(2a)) = i\pi k$ with $k \in \mathbb{Z} \setminus \{0\}$, then E has both critical values mapping to fixed points. Since such maps have no finite asymptotic values, while all critical values are strictly preperiodic, it is well known that all periodic orbits are repelling. In particular, 0 is a repelling fixed point with $E'(0) = k\pi$.

In the rest of the paper, we will need

$$P := \bigcup_{k \geq 0} E^{\circ k}(\{v, v'\}) \quad (\text{the finite postsingular set})$$

and $V := \mathbb{C} \setminus P$; this carries a unique normalized hyperbolic metric.

LEMMA 4 (Dynamic Rays at Critical Values).

For every postsingularly preperiodic map E , there are preperiodic dynamic rays which land at the two critical values, at least one ray at each critical value.

Note that in the special case of $E(z) = k\pi \sinh(z)$, both \mathbb{R}^+ and \mathbb{R}^- are easily seen to be periodic dynamic rays landing at 0; since for these maps, the critical values map to 0, the claim of the lemma is obvious.

PROOF. Choose some periodic $p_0 \in P$ and consider a continuously differentiable curve $\gamma_0: [0, \infty) \rightarrow \mathbb{C} \setminus P$ with $\gamma_0(0) = p_0$ and $\gamma_0(t) \rightarrow \infty$ as $t \rightarrow \infty$. Using this, we construct a family of curves $\gamma_n: [0, \infty) \rightarrow \mathbb{C}$ such that $E(\gamma_{n+1}(t)) = \gamma_n(F(t))$ for all t ; the curve γ_{n+1} is uniquely determined by requiring that $p_{n+1} := \gamma_{n+1}(0)$ is the unique periodic point with $E(p_{n+1}) = p_n$. We claim that there are k, l such that γ_l and γ_{l+k} are homotopic rel P . Observe that homotopies of γ_n lift to homotopies of γ_{n+1} . We will use the hyperbolic metric in $V := \mathbb{C} \setminus P$.

Let C be a circle of large radius ρ , such that C surrounds all of P . Then $E^{-1}(C)$ consists of two unbounded curves L_1 and L_2 , one in the right half plane and one in the left half plane, such that both lines have bounded real parts, while the imaginary parts tend to $\pm\infty$, and both curves are $2\pi i$ -translation invariant. Moreover, all of P is contained in the unique connected component of $\mathbb{C} \setminus (L_1 \cup L_2)$ with bounded real parts. There is a $\delta > 0$ such that every $z \in L_i$ which is surrounded by C can be connected to C by a curve in V of hyperbolic length of at most δ ; see Figure 1.

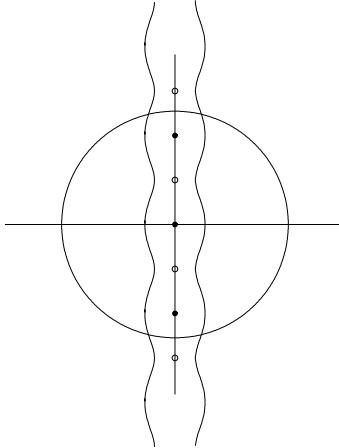


FIGURE 1. Illustration of the proof of Lemma 4 for the map $E(z) = \pi \sinh(z)$. Drawn are the circle C and the two preimage curves L_1 and L_2 . Solid dots indicate the two critical values $\pm i\pi$ as well as their common image, the repelling fixed point 0. Some critical points are indicated by small circles.

Let M be the period of p_0 . Choose ε small enough so that all Euclidean disks $D_\varepsilon(p)$ have disjoint closures for all $p \in P$, and such that $E^{\circ M}(\overline{D_\varepsilon(p_0)}) \supset D_\varepsilon(p_0)$. Let ℓ_0 be the hyperbolic length of γ_0 between $D_\varepsilon(p_0)$ and C . Then the hyperbolic length of γ_1 between $E^{-1}(D_\varepsilon(p_0))$ and some L_i is less than ℓ_0 ; in fact, there is an $\eta < 1$ such that this hyperbolic length is less than $\eta\ell_0$ because of uniform contraction on compact sets of V . After a homotopy of γ_1 we may assume that the hyperbolic length of γ_1 between $E^{-1}(D_\varepsilon(p_0))$ and C is less than $\eta\ell_0 + \delta$, and γ_1 intersects C and $\partial E^{-1}(D_\varepsilon(p_0))$ only once each. After M iterations, it follows that (after an appropriate homotopy) the hyperbolic length of γ_M between C and $D_\varepsilon(p_0)$ is less than $\eta^M\ell_0 + M\delta$. If ℓ_0 is sufficiently large, then $\eta^M\ell_0 + M\delta < \ell_0$; therefore, the hyperbolic lengths of all γ_{kM} (for $k \in \mathbb{N}$) between $D_\varepsilon(p_0)$ and C are uniformly bounded above; but this implies that there are only finitely many homotopy classes available. (For clarity of exposition, this argument has assumed that every curve intersects C and $\partial D_\varepsilon(p_0)$ only once; however, this is only a superficial problem; compare [SZ2]).

Once we know that we have a curve γ_0 which is homotopic to γ_M up to homotopy, we stop applying homotopies and consider each γ_{n+1} as a preimage of γ_n . It then follows that the curves γ_{kM} converge as $k \rightarrow \infty$ to a periodic dynamic ray landing at p_0 ; the details are the same as in [SZ2, Section 6]. The claim follows. \square

Using two preperiodic dynamic rays $g_{\underline{s}}$ and $g'_{\underline{s}}$ landing at the critical values v and v' , we can introduce a dynamical partition as follows: let $U' := \mathbb{C} \setminus (g_{\underline{s}} \cup g'_{\underline{s}} \cup \{v, v'\})$; since U' is simply connected and contains no singular values, it follows that $U := E^{-1}(U')$ consists of countably many connected components W such that $E: W \rightarrow U'$ is a conformal isomorphism for every W . Every critical point has local mapping degree 2, so it is the landing point of exactly two pre-image rays of $g_{\underline{s}}$, $g'_{\underline{s}}$. More precisely, if c is a critical point with $E(c) = v$, say, then c is the landing point of two preimage rays of $g_{\underline{s}}$, such that one preimage ray has real parts tending to $+\infty$ and the other preimage ray has real parts tending to $-\infty$. The analogous fact is true for the critical points c' with $E(c') = v'$. The reason is that critical points are spaced at distances $i\pi$, and every connected component of U must be mapped by E onto U' . Therefore, the dynamical quotient $\mathbb{C}/(2\pi i\mathbb{Z})$ contains exactly two components of U . The connected components of U can thus be described with labels $\mathbf{u} \in \mathbb{Z}/2 = \mathbb{Z} \dot{\cup} (\mathbb{Z} + \frac{1}{2})$ such that $U_{\mathbf{u}+1}$ is the $2\pi i$ -translate of $U_{\mathbf{u}}$ for all $\mathbf{u} \in \mathbb{Z}/2$, and $\partial U_{\mathbf{u}} \cap \partial U_{\mathbf{u}+\frac{1}{2}} \neq \emptyset$ for all \mathbf{u} . This way, we can define *itineraries* for every orbit (z_k) which

stays in U entirely: the itinerary of z_1 is the sequence $u_1 u_2 u_3 \dots$ such that $z_k \in U_{u_k}$ for each k .

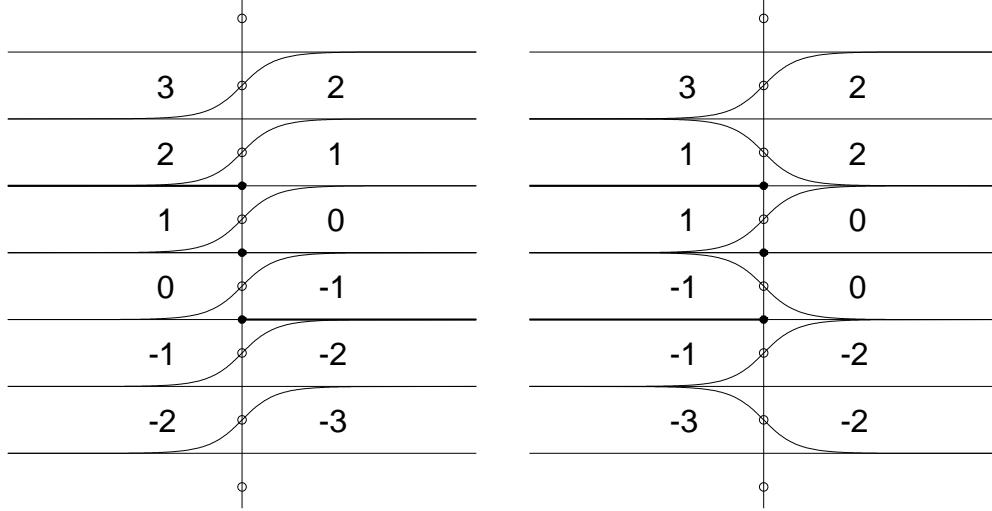


FIGURE 2. The partition of \mathbb{C} by preimages of rays landing at the critical values, together with the labels of the components, shown for the map $E(z) = \pi \sinh(z)$. Solid dots indicate again the two critical values $\pm\pi i$ and their common image 0; small circles indicate the critical points. Heavy lines indicate one dynamic ray landing at each of the two critical values (in this simple case, these rays are horizontal lines); the curved lines are the preimages of these dynamics rays and land at the critical points. The left and right pictures show different choices for the used rays at the critical values, and hence different patterns of the rays landing at the critical points.

By Theorem 1, every dynamic ray g_s has asymptotically constant imaginary parts: $\lim_{t \rightarrow \infty} \operatorname{Im}(g_s(t))$ exists in \mathbb{R} . It follows that every component U_u has bounded height: there is a number $h > 0$ such that for every $u \in \mathbb{Z}/2$ and every $z, w \in U_u$, $|\operatorname{Im}(z) - \operatorname{Im}(w)| < h$. Therefore, knowing the itinerary of an orbit in U means knowing the imaginary parts of this orbit up to additive errors of less than h .

LEMMA 5 (“Almost” Every Point in R).

For every itinerary $u_1 u_2 u_3 \dots$, among all points $z \in \mathbb{C}$ with orbit in U and with itinerary $u_1 u_2 u_3 \dots$, there are at most two which are not in the set R of rays; and if there are two, then both escape.

PROOF. Recall that $V := \mathbb{C} \setminus P$ is the complement of the finite post-critical set and carries a unique normalized hyperbolic metric. Since $E^{-1}(V) \subset V$ is a strict inclusion, every branch of E^{-1} (along any curve of finite length) is a strict contraction with respect to the hyperbolic metric on V (on domain and range). Denote the hyperbolic distance on V between two points $z, w \in V$ by $d_V(z, w)$. Let $\xi > 0$ be such that all $p \in P$ have $|\operatorname{Re}(z)| < \xi$.

Let $w_1 \neq z_1$ be two points with orbits in U and with common itinerary $\underline{u} := u_1 u_2 u_3 \dots$ and suppose that both are not in R . Since E restricted to any U_u is injective, it follows that $z_k \neq w_k$ for all k . Moreover, $|\operatorname{Im}(z_k) - \operatorname{Im}(w_k)| < h$ for every k , where h bounds the height of every U_u .

By Lemma 3, there is a constant $\eta > 0$ with the following property: if there is an index k such that $||\operatorname{Re}(z_k)| - |\operatorname{Re}(w_k)|| > \eta$, then at least one of z_k and w_k is in R , hence $z_1 \in R$ or $w_1 \in R$. Therefore, if z_1 and w_1 do not belong to R , then $||\operatorname{Re}(z_k)| - |\operatorname{Re}(w_k)|| < \eta$ for every k . Therefore, either both orbits z_k and w_k are bounded, or both escape, or both are unbounded without escaping. We treat the three cases separately.

If both orbits are unbounded but not escaping, then they must infinitely often visit some compact set $K \subset V$. If $c := \max\{|a|, |b|\}$, then $|E(z)| \leq c \exp |\operatorname{Re}(z)| + c$, so there is an $x > \xi + \eta + 1$ such that infinitely many z_k satisfy

$$x \leq |\operatorname{Re}(z_k)| \leq ce^x + c.$$

Since always $||\operatorname{Re}(z_k)| - |\operatorname{Re}(w_k)|| < \eta$ and $|\operatorname{Im}(z_k - w_k)| < h$, it follows that for those indices k , $d_V(z_k, w_k)$ are uniformly bounded above by some number $d_x > 0$. But the pull-back steps are contracting, uniformly on the compact set K , and this implies that $z_1 = w_1$, a contradiction.

If both orbits (z_k) and (w_k) escape, then the sequences $|\operatorname{Re}(z_k)|$ and $|\operatorname{Re}(w_k)|$ tend to ∞ in such a way that $||\operatorname{Re}(z_k)| - |\operatorname{Re}(w_k)|| < \eta$. If $|\operatorname{Re}(z_k) - \operatorname{Re}(w_k)| < \eta$ infinitely often, then $|z_k - w_k|$ would be bounded above infinitely often while $|z_k| \rightarrow \infty$, and this would imply that $d_V(z_k, w_k) \rightarrow 0$ at least for a subsequence. But as above, we would then have $z_1 = w_1$, again a contradiction. Therefore, for sufficiently large k the real parts of z_k and w_k always have different signs, and given z_k , there is just one choice for $w_k \neq z_k$. Since the dynamics is injective for the set of points with identical itineraries, there can be at most two escaping orbits which are not on rays and which have the same itinerary.

The remaining case is that both orbits are bounded. To treat this case, observe first that $P \subset U$: all points in P have well-defined itineraries because both critical values are strictly preperiodic and the only non-escaping points on the partition boundaries are the critical points.

Since both critical orbits land on repelling cycles, it is quite easy to see that no point in P shares its itinerary with any other non-escaping point, so none of the two orbits (z_k) and (w_k) can land on P . If at least one of the two orbits, say (w_k) , does not accumulate on P , then there is a compact subset $K \subset V$ with bounded hyperbolic diameter which contains all w_k and infinitely many z_k (because the periodic orbits in P are repelling), so this implies again that $z_1 = w_1$.

The last case is that both orbits accumulate at P . Choose some periodic $p \in P$ in the accumulation set of (z_k) , and $\varepsilon > 0$ such that the disk $D_{2\varepsilon}(p) \subset U$ and $D_{2\varepsilon}(p) \cap P = \{p\}$. There is an $n > 0$ such that all points in $D_\varepsilon(p)$ have common itinerary with p for at least n entries. By choosing $\varepsilon > 0$ sufficiently small, we may assume that n is large enough so that no two points in P have common itineraries for n entries.

Let m be the period of p and choose $\varepsilon' > 0$ such that $E^{\circ m}(\overline{D_{\varepsilon'}(p)}) \subset D_\varepsilon(p)$. Then there are infinitely many k such that $z_k \in D_\varepsilon(p) \setminus D_{\varepsilon'}(p)$. Recall that the orbits (z_k) and (w_k) are bounded. Since bounded $d_V(z_k, w_k)$ would imply $z_1 = w_1$, it follows that for every subsequence $z_{k_l} \in D_\varepsilon(p) \setminus D_{\varepsilon'}(p)$, the corresponding sequence w_{k_l} must converge to P ; more precisely, this sequence must converge to p because this is the only point in P whose itinerary coincides with that of z_{k_l} long enough. Therefore, if M_l is the number of common entries in the itineraries of p and w_{k_l} (and hence of z_{k_l}), then clearly $M_l \rightarrow \infty$. Then there are numbers $M'_l \leq M_l$ such that $w_{k_l+M'_l} = E^{\circ M'_l}(w_{k_l}) \in D_\varepsilon(p) \setminus D_{\varepsilon'}(p)$, and $M'_l \rightarrow \infty$.

Similarly as above, the points $z_{k_l+M'_l}$ must converge to p ; but this is impossible: let $U'' \subset U$ be the set of points whose itineraries coincide with that of p for at least m steps; then $E^{\circ m}: U'' \rightarrow U$ is injective. If we had $z_{k_l+M'_l} = E^{\circ M'_l}(z_{k_l}) \in D_{\varepsilon'}(p)$, then we could pull back M'_l times, and z_{k_l} would have to be very close to a point in P , which is a contradiction. This excludes the case that z_k and w_k are bounded and proves the claim. \square

THEOREM 6 (Dynamic Rays Land).

For every postsingularly preperiodic map E , every dynamic ray lands in \mathbb{C} .

PROOF. Let $g_{\underline{s}}$ be a dynamic ray, which is a curve $g_{\underline{s}}: (t_{\underline{s}}, \infty) \rightarrow \mathbb{C}$ (even a C^∞ curve) with bounded imaginary parts and $\operatorname{Re}(g_{\underline{s}}(t)) = \pm\infty$ as $t \rightarrow \infty$ (Theorem 1). By [RS, Proposition 6.6], $t_{\underline{s}} > 0$ implies that $g_{\underline{s}}$ lands at an escaping point, so we may suppose that $t_{\underline{s}} = 0$.

Let $L_{\underline{s}} \subset \overline{\mathbb{C}}$ be the limit set of $g_{\underline{s}}$: this is the set of all possible limits of $g_{\underline{s}}(t_n)$ as $t_n \searrow 0$. It is well known that $L_{\underline{s}} = \bigcap_{t>0} \overline{g_{\underline{s}}((0, t))}$, which implies that $L_{\underline{s}} \subset \overline{\mathbb{C}}$ is compact and connected.

If $g_{\underline{s}}$ is one of the rays bounding the partition $U_{\mathbf{u}}$, then it lands at a critical point by definition, and there is nothing to show. Otherwise, the entire ray is contained in a single domain $U_{\mathbf{u}}$, hence $L_{\underline{s}} \subset \overline{U_{\mathbf{u}}}$.

Pick any $w \in L_{\underline{s}}$. First we treat the case that $w \in R$, i.e. there are an external address \underline{s}'' and a $t'' > t_{\underline{s}''}$ such that $w = g_{\underline{s}''}(t'')$. By (4), we have the asymptotics $E^{\circ k}(w) = \pm F^{\circ k}(t'') + 2\pi i s_{k+1}'' + O(1)$ as $k \rightarrow \infty$.

Choose any $t' \in (0, t_{\underline{s}''})$. Since $g_{\underline{s}}(t')$ obeys similar asymptotics, there is an $M \in \mathbb{N}$ such that for all $m \geq M$

$$\left| |\operatorname{Re}(E^{\circ m}(w))| - |\operatorname{Re}(E^{\circ m}(g_{\underline{s}}(t')))| \right| > \eta + 1$$

with the constant $\eta > 0$ from Lemma 3 (using the height h of the fundamental domains $U_{\mathbf{u}}$). But we may choose t sufficiently close to 0 so that $g_{\underline{s}}(t)$ is close enough to w such that

$$|E^{\circ M}(g_{\underline{s}}(t)) - E^{\circ M}(w)| < 1,$$

hence

$$\left| |\operatorname{Re}(E^{\circ M}(g_{\underline{s}}(t)))| - |\operatorname{Re}(E^{\circ M}(g_{\underline{s}}(t')))| \right| > \eta.$$

By Lemma 3, this means that $g_{\underline{s}}(t)$ escapes with much greater real parts than $g_{\underline{s}}(t')$; but this contradicts the asymptotics (4) of the two orbits under the condition $t < t'$. This excludes the possibility that w is on a dynamic ray, so $L_{\underline{s}} \subset U_{\mathbf{u}} \cup C \cup \{\infty\}$, where C denotes the set of critical points of E (which are the only boundary points of $U_{\mathbf{u}}$ that are not on rays).

If the point $w \in L_{\underline{s}}$ does not have a well-defined itinerary, then it must be either ∞ or one of the countably many points on the backwards orbits of the two critical values. All other points in $L_{\underline{s}}$ have identical itineraries, but none are on dynamic rays, so by Lemma 5, there can be at most two such points. It follows that $L_{\underline{s}}$ is countable. However, since $L_{\underline{s}}$ is connected, it contains at most one point, so $g_{\underline{s}}$ lands.

The landing point cannot be ∞ : otherwise, there would be potentials $t' > t > 0$ such that $|\operatorname{Re}(g_{\underline{s}}(t))| - |\operatorname{Re}(g_{\underline{s}}(t'))| > \eta$, and again by Lemma 3, this would mean that $g_{\underline{s}}(t)$ escapes with much faster real parts than $g_{\underline{s}}(t')$, again a contradiction. Therefore, $g_{\underline{s}}$ lands at some point in \mathbb{C} . \square

THEOREM 7 (Every Point is Landing Point Or In R).
Every $z \in \mathbb{C}$ is either in R , or the landing point of at least one dynamic ray.

PROOF. We may assume that the itinerary of z is well-defined (or z would eventually map either onto a dynamic ray on the partition boundary, or onto the landing point of such a ray). We may also assume that z does not escape, because every escaping point is either on a ray or the landing points of a ray (Theorem 2).

For $k \geq 1$, let $V_k \subset \mathbb{C}$ be the set of points for which at least the first k entries in their itineraries are well-defined and equal to the itinerary of z . Clearly, $E^{\circ k}: V_k \rightarrow \mathbb{C}$ is a univalent map with connected unbounded image, so each V_k is connected and unbounded as well. Moreover, $V_{k+1} \subset V_k$ implies that the sets $\overline{V}_k \subset \overline{\mathbb{C}}$ form a nested sequence of compact and connected sets containing $\{w, \infty\}$, so $\bigcap_{k \in \mathbb{N}} \overline{V}_k$ is also a compact connected set containing $\{w, \infty\}$.

Each V_k is bounded by finitely many dynamic rays, say at external addresses $\underline{s}^{k,i}$, together with their landing points on the backwards orbits of the critical values $\{v, v'\}$.

If the itinerary of z equals the itinerary of one of the rays $g_{\underline{s}^{k,i}}$, then the itinerary of $E^{\circ k'}(z)$ equals the itinerary of a point in P for sufficiently large k' , hence $E^{\circ k'}(z) \in P$ and z is the landing point of a dynamic ray.

We may thus focus on the case that the itinerary of z differs from the itineraries of all rays $g_{\underline{s}^{k,i}}$; however, the number of common entries in the itineraries of $g_{\underline{s}^{k,i}}$ and z will tend to ∞ as $k \rightarrow \infty$. It now follows that for every ray $g_{\underline{s}^{k,i}}$, there is a $k' > k$ such that $g_{\underline{s}^{k,i}} \cap \overline{V}_{k'} = \emptyset$. For $k \in \mathbb{N}$, let \underline{s}^k be an external address of a dynamic ray in V_k . One can extract a subsequence from the \underline{s}^k which converges pointwise to an external address \underline{s} . This address is necessarily exponentially bounded, and the dynamic ray $g_{\underline{s}}$ has a well-defined itinerary which equals that of z .

By Theorem 6, the ray $g_{\underline{s}}$ lands at some point $z' \in \mathbb{C}$. Then z and z' have identical itineraries, and both are not on rays. Since z does not escape, Lemma 5 implies that $z = z'$. \square

Set $L := \mathbb{C} \setminus R$: the set of landing points of rays. Since the boundary of the partition defining itineraries consists of rays landing at critical points, it follows that every point in L either has a well-defined itinerary, or is on the backwards orbit of the set P .

4. Dynamics and Dimension

It is known from [RS, Sec. 7] that if both critical orbits are strictly preperiodic, then almost every orbit escapes: the set $\mathbb{C} \setminus I$ has Lebesgue measure zero. Recall that the set I consists of the rays together with the landing points of some of the rays, and that R denotes the union of the rays. Surprisingly, the set R has Hausdorff dimension 1 [RS]!

A DIMENSION PARADOX. As a corollary, we have shown the following surprising result: the set R is a set of Hausdorff dimension 1 consisting of uncountably many curves (“rays”), and each ray connects ∞ to a well-defined landing point in $L = \mathbb{C} \setminus R$. Conversely, every $z \in L$ has one or even several rays in R connecting z to ∞ . One might have expected that the set of landing points should be “smaller” than the set of entire rays, but the opposite is the case: not only does $\mathbb{C} \setminus R$ have greater Hausdorff dimension than R , and not only has it positive or even full Lebesgue measure in \mathbb{C} : it is the complement of the 1-dimensional set R ! In other words, we have a partition of \mathbb{C} into an uncountable union of disjoint sets $Y_i = \{z_i\} \cup R_i^1 \cup R_i^2 \cup \dots$, where each component Y_i consists of one point $z_i \in L$ and one or several rays R_i^j landing at z_i . Every ray R_i^j is a one-dimensional curve, and the uncountable union $R = \bigcup_{i,j} R_i^j$ still has dimension 1, but the union $L = \bigcup_i \{z_i\} = \mathbb{C} \setminus R$ is two-dimensional and has full planar Lebesgue measure as the complement of R .

REMARK ON POSSIBLE EXTENSIONS. It seems quite likely that for maps $z \mapsto ae^z + be^{-z}$ in which one or both critical orbits are allowed to be periodic, rather than preperiodic, similar results hold as for our maps: every dynamic ray lands, and every point in the Julia set is the landing point of one or several dynamic rays. However, because of the superattracting cycles the Fatou set would be non-empty and the set of landing points would no longer have full measure (but still positive). Every point would be either in R , a landing point of finitely many rays, or in a superattracting basin. One might wonder whether the precise description of the dynamics would allow for analogs of “pinched disk” models [D] or some kinds of combinatorial or even topological renormalization (although non-continuity of any possible renormalization for exponential maps has recently been shown by Rempe [R], and this argument is likely to apply in our cases too).

While we have shown that all dynamic rays land, we have not discussed whether the landing points depend continuously on the external address. In fact, this depends on the topology used in \mathcal{S} : as a totally ordered space, \mathcal{S} possesses its “order topology” generated by intervals;

with this topology, the landing points do *not* depend continuously on the address because very nearby addresses can still have very different minimal potentials. It is therefore required to endow \mathcal{S} with a topology which depends on more than finitely many first entries; this is a matter of separate discussion.

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